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Weak chaos and metastability in a symplectic system of many long-range-coupled standard maps

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Abstract. We introduce, and numerically study, a system of N symplectically and globally coupled standard maps localized in a $d = 1$ lattice array. The global coupling is modulated through a factor $r^{-\alpha}$, being r the distance between maps. Thus, interactions are *long-range* (nonintegrable) when $0 \leq \alpha \leq 1$, and *short-range* (integrable) when $\alpha > 1$. We verify that the largest Lyapunov exponent λ_M scales as $\lambda_M \propto N^{-\kappa(\alpha)}$, where $\kappa(\alpha)$ is positive when interactions are long-range, yielding *weak chaos* in the thermodynamic limit $N \to \infty$ (hence $\lambda_M \to 0$). In the short-range case, $\kappa(\alpha)$ appears to vanish, and the behaviour corresponds to *strong chaos*. We show that, for certain values of the control parameters of the system, long-lasting metastable states can be present. Their duration t_c scales as $t_c \propto N^{\beta(\alpha)}$, where $\beta(\alpha)$ appears to be numerically in agreement with the following behavior: $\beta > 0$ for $0 \leq \alpha < 1$, and zero for $\alpha \geq 1$. These results are consistent with features typically found in nonextensive statistical mechanics. Moreover, they exhibit strong similarity between the present discrete-time system, and the α-XY Hamiltonian ferromagnetic model.

PACS. 05.20.-y Classical statistical mechanics – 05.45.-a Nonlinear dynamics and chaos – 05.70.Ln Nonequilibrium and irreversible thermodynamics – 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems

1 Introduction

The study of complex systems has been lately one the most active areas of investigation. This multidisciplinary subject has interest in natural sciences as well as in social and artificial systems. In physics, a major area involved is statistical mechanics. It is precisely for various such systems that nonextensive statistical mechanics (NSM) [1] appears to have its applicability. Typical features present in complex systems are long-range interactions, long-term memory, fractal phase-space structure, scale-free network structure, or even combinations of these characteristics. Quite frequently these systems cannot be correctly described by the well-established Boltzmann-Gibbs statistical mechanics (BGSM). Indeed, they often fail to verify its basic assumptions (equiprobability of phase-space occupation [2] and ergodicity [3] at the thermal equilibrium state). NSM generalises the usual BGSM formalism through the definition of the q-entropy $S_q \equiv \frac{1-\int dx\, [p(x)]^q}{q-1}$ (with $S_1 = S_{BG} \equiv -\int dx\, p(x) \ln p(x)$, where BG stands for *Boltzmann-Gibbs*) under simple constraints [4].

This theory has been extensively applied to nonlinear dynamical systems [1,5–7], as they are one of the most widespread and useful ways for modelling complex phenomena. For example, Hamiltonian systems are at the core of physics and consequently their relevance is evident. There has been an important number of results indicating that certain conservative models (e.g., the α-XY model [8,9], also known as Hamiltonian mean field (HMF), the α-Heisenberg model [10,11], the α-Lennard-Jones-like gas [12]) can present a behaviour departing from the one predicted by the BG formalism. For certain classes of initial conditions and parameters, the system is prevented from virtually attaining its expected equilibrium state in finite time when $N \to \infty$. In other words, if the $N \to \infty$

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limit is taken before the $t \to \infty$ limit, the system becomes nonergodic. This nonergodicity is also reflected in a variety of anomalous behaviours such as non-Gaussian momenta probability density functions, negative specific heat, aging, and others [13–16]. On the other hand, other (simpler) nonlinear dynamical systems, as for example maps, have emerged in many contexts, very often exhibiting new and interesting results. Among these systems we find the logistic map, the standard map, coupled map lattices, discretised Lorentz gas, and many others [17]. Both low- and high-dimensional discrete-time dynamical systems have been studied under the framework of NSM [18–20].

It is known quite well that the number of degrees of freedom of a system defines the possibilities that its dynamics may approach. A clear illustration of this fact in low-dimensional systems (e.g., the 2-dimensional Chirikov-Taylor standard map) is the KAM tori, a complex structure in the map phase-space that separates chaotic from regular regions. While much is known for systems with few degrees of freedom, the situation is more intricate when many elements are involved. In addition to this, studying continuous-time many-body systems can turn out to be very difficult, if not impossible at all, due to the considerable computational time needed to integrate the evolution equations. This is even more so when long-range forces are involved, as it is not justified to neglect any interaction between the elements.

An alternative for this problem is the substitution of the continuous-time system for simpler discrete-time systems, such as maps, which conserve many of the important features of the physics involved. This substitution is in fact justified interpreting a map system as the intersection (in the Gibbs Γ phase-space) of a Poincaré plane with the orbit of a higher dimensional Hamiltonian. More precisely, if we consider a *time-independent* Hamiltonian system with $(N+1)$ degrees of freedom ($(2N+2)$-dimensional Γ phase-space), a $(2N)$-dimensional symplectic map is the result of taking a Poincaré section of a constant energy hypersurface [17]. The recurrence time is discrete and the map is useful in displaying various properties of the original Hamiltonian system [21].

The archetypical logistic map (a $D = 1$, dissipative, nonlinear map) is one of the most widely studied dissipative systems. In part due to its simplicity, it is often used to illustrate many of the most important features of chaos. In recent works, Robledo and Baldovin [22] have analytically proved, using standard renormalisation-group techniques, that the dynamics of the logistic map at its critical point is well-described within a NSM frame. The sensitivity to initial conditions is a q-exponential function [23], and is related to the entropy production through a q-generalised Pesin-like identity, linking the sensitivity to initial conditions to the q-entropy S_q with $q = 0.2445\ldots$ Moreover, the logistic-map with noise (a Langevin-like generalisation of the usual logistic map) has been found to present two-step relaxation processes and aging (presenting interesting common points with slow glassy dynamics [24]). Other aspects of the NSM theory were also studied in nonlinear dynamical systems, such as the one-dimensional z-logistic family of maps considering the (multi)fractal nature of the attractor at the edge of chaos [25]. Regarding two-dimensional ($D = 2$) *dissipative* systems (as the Henón map or its linearised version, the Lozi map), results indicate that it presents the same value of q as the logistic map, therefore suggesting a common universality class [26]. Regarding two-dimensional ($D = 2$) *conservative* maps, a very interesting example, the triangle Casati-Prosen map (mixing, ergodic, but with vanishing Lyapunov exponent), has been recently studied in connection to the entropy S_q with $q = 0$ [27].

Moving further into conservative discrete-time systems, two-step relaxation has been also observed at the edge of chaos for the Chirikov-Taylor standard map, a paradigm of 2-dimensional symplectic maps. This map has been thoroughly studied and explained by means of the KAM-theorem [17]. The standard map is known to be area preserving, hence its variables are often considered as "coordinate" and "momentum", in analogy with Hamiltonian systems. Along this lines, it is appealing to look to the more general case of a *symplectic* system of N coupled standard maps ($D = 2N, N \geq 1$). Different efforts were done for values of N as low as $N = 2$ up to $N = 500$ [20,28,29]. In these cases, the same two-step relaxation was found. But in all these cases, the coupling was done through the *momentum* variables. Even though this has its own interest, it would be instructive to see the effects of coupling such a system through the *coordinate* variables, as is the case in more realistic situations.

In this work we will study a high-dimensional globally coupled conservative map system that, as discussed above, presents many of the characteristics of Hamiltonian dynamics. Our purpose is to contribute to the understanding of the role that the concepts of NSM play in the anomalous features present in long-range-interacting dynamical systems. Our results show interesting similarities between this map model and many-body long-range-interacting Hamiltonian dynamics, in particular the α-XY model. In both cases, long-lasting metastable states, as well as weak chaos, are present in the thermodynamic limit. Both features point out that under certain initial conditions, orbits do *not* visit with equal probability the entire Γ phase space, or, in other words, there is a failure in ergodicity.

In the next section we introduce and describe the model, as well as some relevant details of our simulations. In Section 3 we present our numerical results. We report on the scaling behaviour of the largest Lyapunov exponent with the system size N, and point out connections with Hamiltonian dynamics. We analyse the relaxation to equilibrium, and we make a systematic characterisation as a function of the system parameters. Finally, a discussion and summary are presented in Section 4.

2 Model

Our model is a set of N symplectically coupled (hence conservative, see appendix) standard maps, where the coupling is made through the *coordinates* as follows:

$$\begin{aligned} \theta_i(t+1) &= \theta_i(t) + p_i(t+1) \quad (\text{mod } 1), \\ p_i(t+1) &= p_i(t) + \frac{a}{2\pi}\sin[2\pi\theta_i(t)] \\ &\quad + \frac{b}{2\pi\tilde{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{sin[2\pi(\theta_i(t)-\theta_j(t))]}{r_{ij}^{\alpha}} \quad (\text{mod } 1), \end{aligned} \tag{1}$$

where t is the discrete time $t = 1, 2, \ldots$, and $\alpha \geq 0$. The a parameter is the usual nonlinear constant of the individual standard map, whereas the b parameter modulates the overall strength of the long-range coupling. Both parameters contribute to the nonlinearity of the system; it becomes integrable when $a = b = 0$. For simplicity, we have studied only the cases $a > 0$, $b > 0$, but we expect similar results when one or both of these parameters are negative. The systematic study of the whole parameter space is certainly welcome. Notice that, in order to describe a system whose phase space is bounded, we are considering, as usual, only the torus (mod 1). Additionally, the maps are localised in a one dimensional ($d = 1$) regular lattice with periodic boundary conditions. The distance r_{ij} is the minimum distance between maps i and j, hence it can take values from unity to $\frac{N}{2}$ $(\frac{N-1}{2})$ for even (odd) number N of maps. Note that r_{ij} is a fixed quantity that, modulated with the power α, enters equation (1) as an effective time-independent coupling constant. As a consequence, α regulates the range of the interaction between maps. The sum is global (i.e., it includes every pair of maps), so the limiting cases $\alpha = 0$ and $\alpha = \infty$ correspond respectively to infinitely long range and nearest neighbours. In our case $d = 1$, thus $0 \leq \alpha \leq 1$ ($\alpha > 1$) means *long-range* (*short-range*) coupling. Moreover, the coupling term is normalised by the sum [9,30] $\tilde{N} \equiv d\int_1^{N^{1/d}} dr\, r^{d-1}\, r^{-\alpha} = \frac{N^{1-\alpha/d}-\alpha/d}{1-\alpha/d}$, to yield a non-diverging quantity as the system size grows (for simplicity, we have replaced here the exact discrete sum over integer r by its continuous approximation).

For $\alpha = 0$, similar models exist in the literature though in different contexts [19,31,32]. The present particular choice for the coupling was made with the purpose of comparison with many-body Hamiltonian systems. Indeed, one can derive the map set of equation (1) by applying a discretisation procedure to the α-XY model with an external field (for more details see [17,32]). As a consequence of having $N-1$ terms in the coupling summation and the fact that there are N maps, the simulation times are of order $O(N^2)$. For this reason it is a difficult task to numerically simulate equation (1) for large values of N. To overcome this problem, we used an algorithm that takes advantage of the symmetry of the lattice [33] and shortens the simulation time to $O(N \ln N)$.

Initial coordinates and momenta were randomly taken from the following uniform distributions: $\theta_i \in [\theta_0 - \delta\theta, \theta_0 + \delta\theta]$ and $p_i \in [p_0 - \delta p, p_0 + \delta p]$. For the coordinates we used $\theta_0 = \delta\theta = 0.5$ (i.e., $\theta_i \in [0, 1]$, *homogeneous* coordinate initial conditions). For the momenta, we concentrated in two cases. To study the sensibility to initial conditions we used a uniform distribution over the whole phase-space ($p_0 = \delta p = 0.5$). In the analysis of the relaxation to equilibrium we used a thin *waterbag* intial condition [18] with $p_0 = 0.3$ and $\delta p = 0.05$. We also checked *inhomogeneous* initial conditions, in particular $p_0 = \theta_0 = 0.3$ and $\delta p = \delta\theta = 0.05$ (not shown in this work). Note that in this case there is no translational symmetry in the coordinates. For sufficiently long times, our simulations yield the same sensitivity to initial conditions than the ones we obtain for the symmetric case $\theta_0 = \delta\theta = 0.5$. On the other hand, this type of inhomogeneous initial condition in the coordinates has a certain influence in the dependence in time of the relaxation to equilibrium of certain quantities as, for example, the mean kinetic energy. This will be further commented in the next section. We note that the systematic study of the role of initial conditions is very instructive, but it is out of the scope of the present work.

3 Results

3.1 Sensitivity to initial conditions

In order to analyse the sensibility to initial conditions, we numerically studied the largest Lyapunov exponent (LLE) for different values of parameters a, b, α and N. To calculate the LLE, we used the well-known method developed by Benettin et al. [34]. As a consequence of the symplectic structure of equation (1), the Lyapunov spectrum in the $2N$-dimensional phase space of the map is characterised by N pairs of Lyapunov coefficients, where each element of the pair is the negative of the other element. Therefore, the LLE sets an upper bound for the absolute value of the entire spectrum of exponents.

We concentrated on the evolution of the LLE for different values of N starting with $\theta_0 = 0.5$, $\delta\theta = 0.5, p_0 = 0.5$ and $\delta p = 0.5$, $\forall\, \alpha$. In a typical (sufficiently long) realisation, the finite-time LLE, λ_M, is a good estimator for the analytical definition of the LLE (both quantities will coincide when $t \to \infty$ [35]). We averaged between realisations (typically 100) in order to have small statistical fluctuations. We also checked that, for appropriately long times, λ_M does not depend on the initial conditions [36].

In Figure 1 we show the N-dependence of λ_M for typical values of the interaction-range parameter α, and fixed values of parameters $a = 0.005, b = 2$. Our results show that the value of λ_M vanishes with increasing value of N (and consequently, so does the rest of the Lyapunov spectrum) as a power-law $\lambda_M \sim N^{-\kappa(\alpha)}$ for $\alpha \lessapprox 1$, and is a positive constant for $\alpha > 1$ ($\kappa \approx 0$). In the inset we detail κ as a function of α. This shows that, in the large N limit, the map system is *weakly chaotic* for long-range coupling ($\lambda_M \to 0$ when $N \to \infty$), whereas for short-range interactions, λ_M remains positive for all N, meaning *strongly chaotic* dynamics (as expected [37,38]). This result is similar to the one numerically measured and analytically predicted for the α-XY model [39], thus confirming equivalent behaviours. Indeed, and as stated by Anteneodo and Vallejos, this scaling is typical of systems

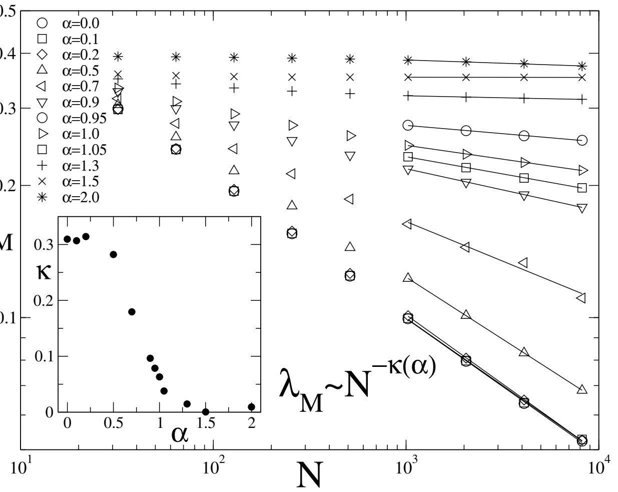

Fig. 1. Lyapunov exponent dependence on system size N in log-log plot, showing that $\lambda_M \sim N^{-\kappa(\alpha)}$. Initial conditions correspond to $\theta_0 = 0.5$, $\delta\theta = 0.5$, $p_0 = 0.5$ and $\delta p = 0.5$. Fixed parameters are $a = 0.005$ and $b = 2$. We averaged over 100 realisations. *Inset:* κ vs. α, exhibiting weak chaos in the limit $N \to \infty$ when $0 \leq \alpha \lessapprox 1$.

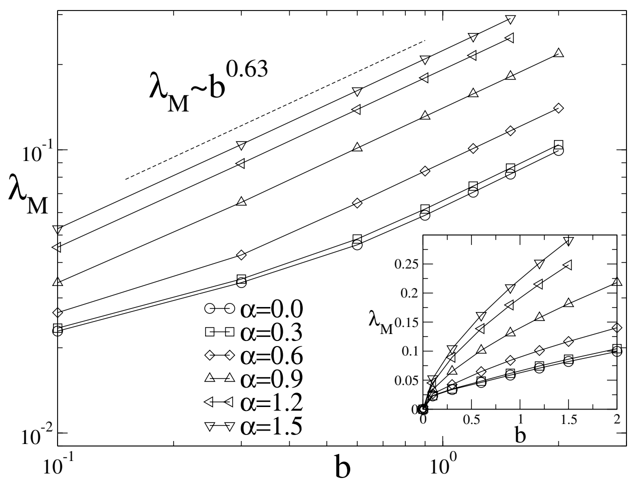

Fig. 3. Lyapunov exponent dependence on b in log-log plot. Fixed constants are $N = 1024$ and $a = 0.005$. Initial conditions correspond to $\theta_0 = 0.5$, $\delta\theta = 0.5$, $p_0 = 0.5$ and $\delta p = 0.5$. We averaged over 100 realisations. *Inset*: same data in linear-linear plot.

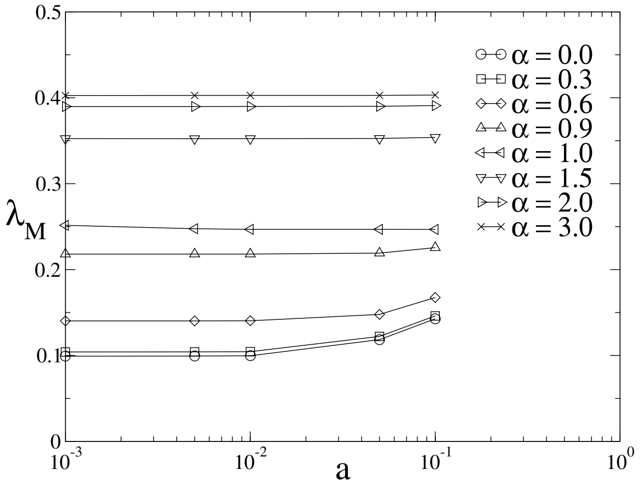

Fig. 2. Lyapunov exponent dependence on a for different values of α. Fixed constants are $N = 1024$ and $b = 2$. Initial conditions correspond to $\theta_0 = 0.5$, $\delta\theta = 0.5$, $p_0 = 0.5$ and $\delta p = 0.5$. We averaged over 100 realisations.

with couplings of the form $1/r^\alpha$ [40]. Preliminary simulations suggest that the fact that the weak chaos region extends slightly over $\alpha = 1$ is an expected consequence of both finite-size and finite-time effects.

The dependence of λ_M with the nonlinear parameter a for different ranges of the interaction α is shown in Figure 2. We can see that, for $\alpha < 1$, λ_M decreases with a and saturates for $a \ll 1$. This illustrates the influence of this nonlinear term. For increasing a, the sensibility to initial conditions raises. On the other hand a has almost no effect when $\alpha > 1$, where the λ_M is approximately constant on the whole a-range.

In Figure 3 we exhibit the dependence on the coupling parameter b, also varying α. For high values of b ($b \gg 1$), $\lambda_M = c(\alpha) b^{0.63}$ $\forall\alpha$, being $c(\alpha)$ a nonlinear function. Preliminary simulations show that this exponent varies only slightly with N for low values of a. For $a = 0.005$, in the case $b = 0$, we verified that $\lambda_M = 0$, $\forall\alpha$. For $\alpha > 1$ this scaling is valid even for $b \ll 1$. On the other hand, if $\alpha < 1$, a deviation from this power-law behaviour emerges when $b \ll 1$.

Our characterisation illustrates the fact that, quite generically, the sensibility to initial conditions is strongly governed by the range of the interactions. Orbits are clearly chaotic in the short-range case (as expected) and strong numerical evidence of vanishing LLE emerges when interactions are long-range, revealing a weakly chaotic regime.

Our present results for the maximal Lyapunov exponent can be approximatively summarised as follows:

$$\lambda_M(a, b, N, \alpha) \propto f_\alpha(b)\, N^{-\kappa(\alpha)}, \tag{2}$$

where $0 \leq a \ll 1$; $0 \leq b \lessapprox 2$; $N \gg 1$, $\alpha \geq 0$, and $f_\alpha(b)$ is some function of (b, α) (e.g., $f_\alpha(b) \propto b^{0.63}$ for $\alpha > 1$).

It is worth to make a note about the synchronisation behaviour of our model. The system does not synchronise for the range of parameters a and b chosen throughout this work (i.e., the order parameter is approximately zero). All our results correspond to the incoherent regime. On the other hand, the system presents coherent behaviour for certain (trivial) initial conditions (e.g. $\delta p = \delta\theta = 0$), and for similarly trivial parameter values ($p_0 = 0.5, \delta p \ll 1, \theta_0 = 0, \delta\theta \ll 1, a \approx 1, b \ll 1$). There is strong interest on its own in determining if there is a set of nontrivial values of the parameters that exhibit coherent behaviour. Moreover, it is interesting to ask if the results presented here are valid also in the synchronisation regime. An effort in this line would certainly be welcome.

3.2 Temperature evolution

As stated above, system (1) is symplectic, hence (hyper) volume preserving, like Hamiltonian systems. A consequence of this is that θ may be interpreted as a "coordinate" variable and p as the conjugate "momentum". We may define a concept analogous to a *temperature* as twice the mean "kinetic energy" per particle [19,29],

$$T(t) \equiv \left\langle \frac{1}{N} \sum_{i=1}^{N} (p_i(t) - \bar{p}(t))^2 \right\rangle, \tag{3}$$

where $\bar{p}(t) = \frac{1}{N} \sum_{i=1}^{N} p_i(t)$ and $\langle ... \rangle$ denotes ensemble average. This quantity can be interpreted as a dynamical analog, and plays a role similar to the physical temperature. Its interesting to note that the use of (mod 1) for the momenta in equation (1) presents a discontinuous effect that may have an influence on the evolution of $T(t)$. In this sense, we checked an alternative definition of equation (3) [41] that avoids such discontinuity. The results obtained coincide qualitatively to the ones associated with equation (3). For simplicity, we remain with the definition presented above.

We refer to the temperature associated with a *uniform* ensemble distribution in phase space as *BG-temperature*, T_{BG}. This quantity may be analytically calculated,

$$T_{BG} \equiv \frac{1}{N} \sum_{i=1}^{N} \left[\int_0^1 dp_i\, p_i^2 - \left(\int_0^1 dp_i\, p_i \right)^2 \right], \tag{4}$$

which yields $T_{BG} = 1/12 \simeq 0.083$ ($\forall N$). We studied the evolution of the dynamical temperature T for typical values of the parameters as described above, focusing on the relaxation of temperature $T(t)$ towards T_{BG}. We used *waterbag* initial conditions with *homogeneous* coordinate distribution $\theta_i \in [0, 1]$, and momenta centered at $p_0 = 0.3$ with width $\delta p = 0.05$. The manner in which the relaxation takes place depends strongly on the initial conditions. For example, using $0 < p_0 \ll 1$ makes the system to temporarily reach a temperature greater than T_{BG}, as a consequence of the periodic boundary conditions (torus (mod 1)). In general, different initial conditions produce different temperature evolutions, but in all cases the relaxation scaling with N remains similar. We also checked *inhomogeneous* coordinate distributions as initial conditions ($\theta_0 = 0.3$, $\delta\theta = 0.05$), and obtained results qualitatively similar to those corresponding to the homogeneous case we present here. Although the temperature value in the metastable state is not as low as in the homogeneous case, long-lasting metastable plateaux appear (very much as in the $\alpha = 0$ case of the α-XY model for initial magnetisation $M = 1$ [42]). The duration of these plateaux also scales with N in a qualitatively similar manner as for the homogeneous case. The particular choice of initial conditions used in this work yields a rather smooth relaxation with, in particular, only one inflexion point, thus simplifying our analysis. This scenario is consistent with the conjecture advanced in [43], namely that, for some properties, the limits $N \to \infty$ and $t \to \infty$ are not interchangeable.

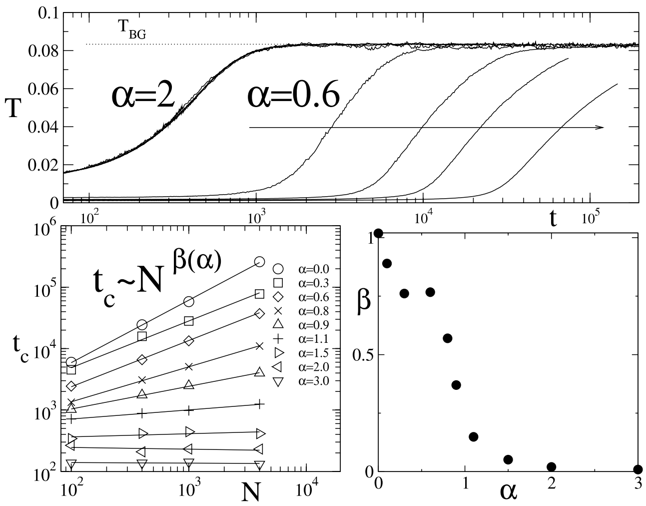

Fig. 4. *Upper panel*: temperature evolution for $\alpha = 2$ and $\alpha = 0.6$ and four system sizes $N = 100, 400, 1000, 4000$. Initial conditions correspond to $\theta_0 = 0.5$, $\delta\theta = 0.5$, $p_0 = 0.3$ and $\delta p = 0.05$. Fixed constants are $a = 0.05$ and $b = 2$. We averaged over 100 realisations. For $\alpha = 2$ the four curves coincide almost completely, all having a very fast relaxation to T_{BG}. For $\alpha = 0.6$ the same sizes are shown, growing in the direction of the arrow. *Left bottom panel*: crossover time t_c vs. N, showing a power-law dependence $t_c \sim N^{\beta(\alpha)}$ with $\beta(\alpha) \geq 0$. *Right bottom panel*: β vs. α shows that for long-range interactions the QS state life-time diverges in the thermodynamic limit. Note that when $\alpha = 0$, $\beta = 1$, and hence $t_c \propto N$. Given the non-neglectable error bars due to finite size effects, the relation $\beta = 1 - \alpha$ is not excluded as possibly being the exact one for $0 \leq \alpha < 1$ ($\beta = 0$ otherwise); more precisely, it is not unplausible that $t_c \propto \frac{N^{1-\alpha}-1}{1-\alpha}$ (hence, for $\alpha = 1$, $t_c \propto \ln N$).

This type of relaxation has already been reported for the particular case $\alpha = 0$ [19]. A two-step process appears: firstly a stage where $T < T_{BG}$, and then a final relaxation to the predicted temperature T_{BG}. The initial regime varies very slowly in time yielding quasistationary (QS) states.

In Figure 4 we show the temperature evolution for $\alpha = 0.6$ and $\alpha = 2$ for sizes $N = 100, 400, 1000, 4000$. We chose $a = 0.05$ and $b = 2$ for convenience, to be able to see the relaxation to equilibrium in a reasonable computational time. The curves crossed by the arrow correspond to the $\alpha = 0.6$ case, being the first one (from left to right) $N = 100$. All four curves for $\alpha = 2$ relax approximately at the same time, so it has the appearance of a single curve. For $\alpha = 0.6$ the typical two-step relaxation is obtained. From now on, we define the crossover time t_c from the QS state to the BG equilibrium state by means of the inflexion point, i.e. the time that corresponds to a maximum in the time derivative of T. The dependence of t_c with N for this choice of parameters and initial conditions is reported in the bottom left panel. The crossover time scales as $t_c \sim N^{\beta(\alpha)}$ $\forall \alpha$. For $\alpha \gtrapprox 1$, $\beta(\alpha) \approx 0$ and then t_c remains constant (as depicted for $\alpha = 2$ in Fig. 4). For

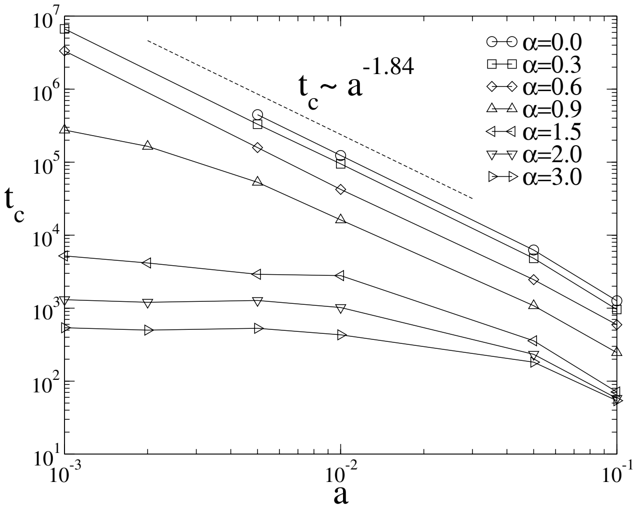

Fig. 5. Temperature dependence on a. Fixed constants are $N = 100$ and $b = 2$. Initial conditions correspond to $\theta_0 = 0.5$, $\delta\theta = 0.5$, $p_0 = 0.3$ and $\delta p = 0.05$. We averaged over 100 realisations.

$\alpha \lessapprox 1$, $\beta(\alpha) > 0$, i.e., t_c diverges in the thermodynamic limit $N \to \infty$. This result indicates, for long-range interactions, the inequivalence of the limits $t \to \infty$, $N \to \infty$, and $N \to \infty$, $t \to \infty$. Consistently, these QS states become permanent (and therefore particularly relevant) when $N \to \infty$. Once again, this situation coincides with that of the α-XY model [9].

Finally, we studied the dependence of t_c with the nonlinear parameter a, for different values of α, fixed coupling parameter $b = 2$, and size $N = 100$. In this case, as well as in the other situations analysed in this work, different behaviours are reported for short- and long-range interactions. For $\alpha > 1$, the value of t_c, in the limit $a \to 0$, tends to a finite value. The situation is different in the long-range case, where, for $\alpha \ll 1$, the crossover time t_c scales as $t_c = d(\alpha)\, a^{1.84}$ being $d(\alpha)$ a nonlinear function. Preliminary calculations show that the deviation for the case $\alpha = 0.9$ and $a \ll 1$ is due to finite-size effects. This scaling law implies that, for long-range interactions and vanishing nonlinear parameter a, the system stays in the metastable regime permanently in the thermodynamical limit.

Our present results for the crossover time t_c can be approximatively summarised as follows:

$$t_c(a, b, N, \alpha) \propto a^{-1.84} g_\alpha(b)\, N^{\beta(\alpha)}, \tag{5}$$

where $0.001 \le a \ll 0.1$, $0 \le b \lessapprox 2$, $N \gg 1$, $0 \le \alpha < 1$, and $g_\alpha(b)$ is some nonlinear function of (b, α).

4 Conclusions

In this work we have analysed a system of N symplectically and globally coupled standard maps, for both short- and long-range interactions. Our results connect with studies made in long-range Hamiltonian systems as well as with other map systems with vanishing largest Lyapunov exponent. We studied a suitably defined dynamical temperature for a region of the parameter space, and report the appearance of long-lasting quasistationary states, followed by a relaxation to the predicted equilibrium value. The relaxation time scales as $t_c \sim N^{\beta(\alpha)}$, with $\beta(\alpha) \approx 0$ when $\alpha \gtrapprox 1$, and $\beta(\alpha) > 0$ when $0 \le \alpha \lessapprox 1$, thus diverging as $N \to \infty$. This is a feature also present in the α-XY Hamiltonian model. This fact, together with a vanishing LLE in the $N \to \infty$ limit, appear to be important ingredients in those systems to which NSM is applicable. Regarding the sensitivity to initial conditions, we calculated the maximum Lyapunov exponent λ_M as a function of the different system parameters, namely, the size of the system N, the nonlinear parameter a and the coupling constant b. We found that, as the number N of maps grows, λ_M vanishes as $N^{-\kappa(\alpha)}$ with $\kappa(\alpha) > 0$ for $0 \le \alpha \lessapprox 1$, and $\kappa \approx 0$ for $\alpha \gtrapprox 1$. The dependence of κ with α is reported and compared to be the same as in the α-XY model. We note that, even though the initial evolution of λ_M depends strongly in the initial conditions (as the temperature does), this scaling does not. In fact, it coincides with that calculated with uniform initial conditions (i.e., initial p_i, θ_i uniformly distributed in the $[0, 1]$ interval) for $t \gg 1$. Our results exhibit that, in the presence of long-range interactions, the system tends to be *weakly chaotic* in the thermodynamic limit. The lack of ergodicity exhibited in our model is thought to be related to a (multi)fractal constraint in the available phase-space that is enhanced in certain limits such as $N \to \infty$. The similarities of the present coupled map model and the α-XY Hamiltonian model suggest that both share features that may have a common dynamical behaviour.

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Appendix A: Symplecticity of the model

If $G(\bar{x})$ denotes a map system, then G is symplectic when its Jacobian $\partial G/\partial \bar{x}$ satisfies the relation [17]:

$$\left(\frac{\partial G}{\partial \bar{x}}\right)^T J \left(\frac{\partial G}{\partial \bar{x}}\right) = J, \tag{A-1}$$

where the superindex T indicates the transposed matrix, and J is the Poisson matrix, defined by

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{A-2}$$

I being the $N \times N$ identity matrix. A consequence of equation (A-1) is that the Jacobian determinant $|\partial G/\partial \bar{x}| = 1$, indicating that the map G is *(hyper)volume-preserving.* In particular, for our model

$$\frac{\partial G}{\partial \bar{x}} = \begin{pmatrix} I & I \\ B & (I+B) \end{pmatrix}, \tag{A-3}$$

where $\bar{x}$ is the $2N$-dimensional vector $\bar{x} \equiv (\bar{p}, \bar{\theta})$, and

$$B = \begin{pmatrix} K_{\theta_1} & c_{21} & \dots & c_{N1} \\ c_{12} & K_{\theta_2} & \dots & c_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1N} & c_{2N} & \dots & K_{\theta_N} \end{pmatrix}, \tag{A-4}$$

with

$$K_{\theta_i} = a\cos[2\pi\theta_i(t)] + \frac{b}{\tilde{N}} \sum_{j\neq i} \frac{cos[2\pi(\theta_i(t)-\theta_j(t))]}{r_{ij}^{\alpha}},$$

and

$$c_{ij} = c_{ji} = -\frac{b}{\tilde{N}} \frac{cos[2\pi(\theta_i(t)-\theta_j(t))]}{r_{ij}^{\alpha}},$$

where $i, j = 1, \dots, N$. It can be seen that,

$$\left(\frac{\partial G}{\partial \bar{x}}\right)^T = \begin{pmatrix} I & B \\ I & (I+B) \end{pmatrix}, \tag{A-5}$$

hence

$$\left(\frac{\partial G}{\partial \bar{x}}\right)^T J = \begin{pmatrix} -B & I \\ -(I+B) & I \end{pmatrix}. \tag{A-6}$$

This quantity, multiplied (from the right side) by the matrix (A-3) yields J. Therefore our system is symplectic. Consequently, the $2N$ Lyapunov exponents $\lambda_1 \equiv \lambda_M$, $\lambda_2, \lambda_3, ..., \lambda_{2N}$ are coupled two by two as follows: $\lambda_1 = -\lambda_{2N} \geq \lambda_2 = -\lambda_{2N-1} \geq ... \geq \lambda_N = -\lambda_{N+1} \geq 0$. In other words, as a function of time, an infinitely small *length* typically diverges as $e^{\lambda_1 t}$, an infinitely small *area* diverges as $e^{(\lambda_1+\lambda_2)t}$, an infinitely small *volume* diverges as $e^{(\lambda_1+\lambda_2+\lambda_3)t}$, an infinitely small N-dimensional *hypervolume* diverges as $e^{(\sum_{i=1}^{N}\lambda_i)t}$ ($\sum_{i=1}^{N}\lambda_i$ being in fact equal to the Kolmogorov-Sinai entropy rate, in agreement with the Pesin identity), an infinitely small $(N+1)$-hypervolume diverges as $e^{(\sum_{i=1}^{N-1}\lambda_i)t}$, and so on. For example, a $(2N-1)$-hypervolume diverges as $e^{\lambda_1 t}$, and finally a $2N$-hypervolume remains constant, thus recovering the conservative nature of the system (of course, this corresponds to the Liouville theorem in classical Hamiltonian dynamics).

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